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Bounds on energy flux for finite energy turbulence

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For incompressible three-dimensional (two-dimensional) turbulence of finite energy, bounds are obtained on energy (enstrophy) flux. To estimate the nonlinear terms, we use a decomposition of the Fourier space into shells of exponentially increasing radii and the property of boundedness in position space of square-integrable functions with Fourier transforms of compact support. In the limit of zero viscosity, it is shown that the three-dimensional (two-dimensional) energy (enstrophy) inertial range, if it exists, cannot have an energy spectrum steeper than $k^{-\frac{8}{3}}$ (k^{-4}). Similar results are obtained for the advection of a passive scalar. The connexion with the problem of homogeneous turbulence and intermittency is briefly discussed.

1. Introduction

It appears to be difficult to derive exact bounds for statistical quantities directly from the Navier–Stokes equations or from the equivalent hierarchy of moment (or cumulant) equations (see, however, Howard 1972). The main reason is that high-order moments (or cumulants) are not bounded from *above* by a suitable combination of lower-order moments but only from *below*. For example, if m is a real centred random variable, $E\{m^4\} \geqslant E\{m^2\}^2$. The situation is much better if, instead of expectation values, we take space integrals. For example, if f is a real function which is square-integrable and has a square-integrable gradient, we have (Ladyzhenskaya 1969, pp. 8, 9)

$$\int_{\mathbb{R}^D} f^4 dx \leqslant C\left[\int_{\mathbb{R}^D} f^2 dx\right]^\alpha \left[\int_{\mathbb{R}^D} (\nabla f)^2 dx\right]^\beta,$$

$$\alpha = 1, \quad \beta = 1 \quad \text{in two dimensions,}$$

$$\alpha = \tfrac{1}{2}, \quad \beta = \tfrac{3}{2} \quad \text{in three dimensions.}$$

Obviously, if we want to use such inequalities, we must consider finite energy turbulence and thus give up homogeneity.

In this paper, we shall obtain bounds for the energy (or enstrophy) flux through a given wavenumber k for finite energy turbulence in the absence of boundaries. From these, we derive bounds on the power law for the energy spectrum in the energy (enstrophy) inertial range. It must be stressed that (i) the treatment is not probabilistic, space integrals being taken instead of expectation values; and (ii) the results are based entirely on a kinematic analysis of the nonlinear terms; problems of existence and uniqueness are not considered:

for such questions, we refer the reader to Lions' (1969) book and to the proofs of smoothness of the solution of the three-dimensional Euler equation during a finite time (Lichtenstein 1925; Ebin & Marsden 1970; Kato 1972; Bourguignon & Brezis 1974; Foias, Frisch & Temam 1975; Temam 1975).

2. Shell decomposition of Fourier space

Here $|\ |$ denotes the Euclidian norm (of a vector) in $\mathbb{R}^D$ (D = space dimension), $\|\ \|$ the $L^2(\mathbb{R}^D)$ norm and $(\ ,\ )$ the $L^2(\mathbb{R}^D)$ scalar product. Let $\{k_n\}$ be the sequence $k_{-1} = 0$, $k_n = 2^n k_0$ $(n \geqslant 0)$, where k_0 is a reference wavenumber. We define the shells

$$S_n = \{\mathbf{k}\,|\,\mathbf{k} \in \mathbb{R}^D, k_{n-1} \leqslant |\mathbf{k}| < k_n\} \quad (n \geqslant 0)$$

and the function spaces

$$\mathscr{S}_n = \{u\,|\,u \in L^2(\mathbb{R}^D), \text{ support of } \hat{u} \subset S_n\}, \quad (n \geqslant 0),$$

where $\hat{u}$ denotes the D-dimensional Fourier transform of u. Clearly $L^2(\mathbb{R}^D)$ is the direct Hilbert sum of the $\mathscr{S}_n$'s $(n \geqslant 0)$ and

$$u = \sum_{n \geqslant 0} u_n,$$

where $u_n = P_n u$ is the orthogonal projection of u on $\mathscr{S}_n$. Notice that the Fourier transform of the operator P_n is simply its product with the characteristic function $\theta_n(\mathbf{k})$ of S_n.

The main properties of this shell decomposition are ($P1$) that space derivatives in $\mathscr{S}_n$ are bounded operators, i.e.

$$\|\partial u_n/\partial x_i\| \leqslant k_n \|u_n\| \quad \text{(a trivial consequence of the definition)},$$

and ($P2$) that the functions $u_n(\ )$ are bounded in position space for any n, i.e.

$$\sup_{x \in \mathbb{R}^D} |u_n(\mathbf{x})| \leqslant C^{st} k_n^{\frac{1}{2}D} \|u_n\|.$$

Proof

$$u_n(\mathbf{x}) = \int_{\mathbb{R}^D} \exp(-i\mathbf{k}.\mathbf{x})\, u_n(\mathbf{k})\, d^D k,$$

$$|u_n(\mathbf{x}) \leqslant \int_{S_n} |u_n(\mathbf{k})|\, d^D k \leqslant \left[\int_{S_n} 1\, d^D k\right]^{\frac{1}{2}} \left[\int_{S_n} |u_n(k)|^2 d^D k\right]^{\frac{1}{2}} \leqslant c k_n^{\frac{1}{2}D} \|u_n\|.$$

Three-dimensional turbulent flows

The velocity field obeys the Navier–Stokes equations

$$\left.\begin{aligned} \partial \mathbf{v}/\partial t + (\mathbf{v}.\nabla)\,\mathbf{v} &= -\nabla p + \nu \nabla \mathbf{v}, \\ \nabla.\mathbf{v} &= 0. \end{aligned}\right\} \tag{1}$$

Projected onto $\mathscr{S}_n$ space, they read

$$\left.\begin{aligned} \partial \mathbf{v}_l/\partial t + P_l((\mathbf{v}.\nabla)\,\mathbf{v}) - \nu \Delta \mathbf{v}_l &= -\nabla p_l, \\ \nabla.\mathbf{v}_l &= 0, \end{aligned}\right\} \tag{2}$$

with $\mathbf{v}_l = P_l\mathbf{v}$ and $p_l = P_l p$. We define $E_n \equiv \frac{1}{2}\|\mathbf{v}_n\|^2$, the energy contained in the shell S_n. The negative rate of change of the energy contained in the first L shells,

$$-\frac{d}{dt}\sum_{l=0}^{L} E_l,$$

is made up of two parts.

(i) A viscous contribution

$$\nu \sum_{l=0}^{L} \|\operatorname{curl} \mathbf{v}_n\|^2,$$

bounded by $\nu k_L^2 \|\mathbf{v}\|^2$, which for finite L converges to zero as $\nu \to 0$.

(ii) A nonlinear contribution which is the energy flux through the wavenumber k_L:

$$\Pi_{\text{energy}}(k_L) = \sum_{l=0}^{L}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{l,m,n}, \tag{3}$$

where
$$b_{l,m,n} = (\mathbf{v}_l, (\mathbf{v}_m \cdot \nabla)\mathbf{v}_n).$$

We may decompose the summation in (3) into four parts:

$$\sum_{l=0}^{L}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} = \sum_{l=0}^{L}\sum_{m=0}^{L}\sum_{n=0}^{L} + \sum_{l=0}^{L}\sum_{m=L+1}^{\infty}\sum_{n=0}^{L} + \sum_{l=0}^{L}\sum_{m=0}^{L}\sum_{n=L+1}^{\infty} + \sum_{l=0}^{L}\sum_{m=L+1}^{\infty}\sum_{n=L+1}^{\infty}.$$

The skew symmetry of $b_{l,m,n}$ $(b_{l,m,n} = -b_{n,m,l})$ gives

$$\Pi_{\text{energy}}(k_L) = \sum_{l=L+1}^{\infty}\sum_{m=0}^{L}\sum_{n=0}^{L} b_{l,m,n} - \sum_{l=0}^{L}\sum_{m=L+1}^{\infty}\sum_{n=L+1}^{\infty} b_{l,m,n}. \tag{4}$$

Since $b_{l,m,n} = (\hat{\mathbf{v}}_l, \widehat{(\mathbf{v}_m \cdot \nabla)\mathbf{v}_n})$ is zero for values of l, m and n such that no triangle can be formed with three wave vectors lying in S_l, S_m and S_n (support condition), we can write

$$\Pi_{\text{energy}}(k_L) = \left(\sum_{n=0}^{L} b_{L+1,L,n} + \sum_{m=0}^{L-1} b_{L+1,m,L}\right) - \sum_{l=0}^{L}\left[\sum_{m=L+1}^{\infty}(b_{l,m,m} + b_{l,m,m+1}) + \sum_{m=L+2}^{\infty} b_{l,m,m-1}\right]. \tag{5}$$

We now seek upper bounds for the $|b_{l,m,n}|$. Denoting vector components by superscripts, $|b_{l,m,n}|$ is expressed as

$$|b_{l,m,n}| = \sum_{i=1}^{3}\sum_{j=1}^{3}\left|\int_{\mathbb{R}^D} v_l^i(\mathbf{x})\, v_m^j(\mathbf{x})\frac{\partial}{\partial x_j} v_m^i(\mathbf{x})\, d^D x\right|$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\mathbb{R}^D}\left|\left(\frac{\partial}{dx_j} v_l^i(\mathbf{x})\right) v_m^j(\mathbf{x})\right| v_n^i(\mathbf{x})\,|\,d^D x.$$

Using the first definition if $l > n$ and the second if $l < n$, we obtain by means of properties $P1$ and $P2$

$$|\mathbf{b}_{l,m,n}| \leqslant \begin{cases} C^{st} k_n\, k_{\inf(m,n)}^{\frac{3}{2}}\, E_l^{\frac{1}{2}} E_m^{\frac{1}{2}} E_n^{\frac{1}{2}} & (l > n), \\ C^{st} k_l^{\frac{5}{2}} E_l^{\frac{1}{2}} E_m^{\frac{1}{2}} E_n^{\frac{1}{2}} & (l < n). \end{cases}$$

Finally we get an upper bound on the energy flux:

$$|\Pi_{\text{energy}}(k_L)| \leqslant C\left\{E_L^{\frac{1}{2}} E_{L+1}^{\frac{1}{2}}\left(\sum_{n=0}^{L} k_n^{\frac{5}{2}} E_n^{\frac{1}{2}}\right) + k_L E_L^{\frac{1}{2}} E_{L+1}^{\frac{1}{2}}\left(\sum_{m=0}^{L-1} k_m^{\frac{3}{2}} E_m^{\frac{1}{2}}\right)\right.$$
$$\left. + \left(\sum_{l=0}^{L} k_l^{\frac{5}{2}} E_l^{\frac{1}{2}}\right)\left[E_{L+1} + E_{L+1}^{\frac{1}{2}} E_{L+2}^{\frac{1}{2}} + \sum_{M=L+2}^{\infty} E_m^{\frac{1}{2}}\left(E_{m-1}^{\frac{1}{2}} + E_m^{\frac{1}{2}} + E_{m+1}^{\frac{1}{2}}\right)\right]\right\}. \quad (6)$$

If we assume $E_l \leqslant Ck_l^{-\alpha}$, we find that

$$\lim_{L\to\infty} \Pi_{\text{energy}}(k_L) = 0 \quad \text{provided that } \alpha > \tfrac{5}{3}.$$

Two-dimensional turbulent flows

The purely vertical vorticity $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$ satisfies

$$\left.\begin{aligned} \partial\boldsymbol{\omega}/\partial t + (\mathbf{v}.\nabla)\boldsymbol{\omega} &= \nu\Delta\boldsymbol{\omega}, \\ \nabla.\mathbf{v} &= 0. \end{aligned}\right\} \quad (7)$$

We define $\Omega_l = \|\boldsymbol{\omega}_l\|^2$ (notice that $\Omega_l \leqslant k_l^2 E_l$) and $\overline{\mathbf{b}}_{l,m,n} = (\boldsymbol{\omega}_l, (\mathbf{v}_m.\nabla)\boldsymbol{\omega}_n)$. For the enstrophy flux $\Pi_{\text{enst}}(k_L)$, the contribution of nonlinear terms to

$$-\frac{d}{dt}\sum_{l=0}^{L} \Omega_l,$$

we have as before

$$|\Pi_{\text{enst}}(k_L)| \leqslant C^{st}\left\{E_L^{\frac{1}{2}} \Omega_{L+1}^{\frac{1}{2}}\left(\sum_{n=0}^{L} k_n^{\frac{5}{2}} \Omega_n^{\frac{1}{2}}\right) + k_L \Omega_L^{\frac{1}{2}} \Omega_{L+1}^{\frac{1}{2}}\left(\sum_{m=0}^{L-1} k_m^{\frac{3}{2}} E_m^{\frac{1}{2}}\right)\right.$$
$$\left. + \left(\sum_{l=0}^{L} k_l^{\frac{5}{2}} \Omega_l^{\frac{1}{2}}\right)\left[E_{L+1}^{\frac{1}{2}} \Omega_{L+1}^{\frac{1}{2}} + E_{L+1}^{\frac{1}{2}} \Omega_{L+2}^{\frac{1}{4}} + \sum_{m=L+2}^{\infty} E_m^{\frac{1}{2}}\left(\Omega_{m-1}^{\frac{1}{2}} + \Omega_m^{\frac{1}{2}} + \Omega_{m+1}^{\frac{1}{2}}\right)\right]\right\}. \quad (8)$$

Now, assuming $E_l \leqslant Ck_l^{-\alpha}$, we find that

$$\lim_{L\to\infty} \Pi_{\text{enst}}(k_L) = 0 \quad \text{provided that } \alpha > 3.$$

Advection of a passive scalar

The passive-scalar density satisfies the equation

$$\partial\phi/\partial t + (\mathbf{v}.\nabla)\phi = \kappa\Delta\phi, \quad (9)$$

where $\mathbf{v}$ is the solution of the Navier–Stokes equations. We define $\Psi_l = \|\phi_l\|^2$ and $b_{l,m,n}^s = (\phi_l, (\mathbf{v}_m.\nabla)\phi_n)$. For the intensity flux $\Pi_{\text{scalar}}(\mathbf{k}_L)$, the contribution of the advection term to

$$-\frac{d}{dt}\sum_{l=0}^{L} \Psi_l,$$

we get as before

$$|\Pi_{\text{scalar}}(\mathbf{k}_L)| \leqslant C^{st}\left\{E_L^{\frac{1}{2}} \Psi_{L+1}^{\frac{1}{2}}\left(\sum_{n=0}^{L} k_n^{\frac{5}{2}} \Psi_n^{\frac{1}{2}}\right) + k_L \Psi_L^{\frac{1}{2}} \Psi_{L+1}^{\frac{1}{2}}\left(\sum_{m=0}^{L-1} k_m^{\frac{3}{2}} E_m^{\frac{1}{2}}\right)\right.$$
$$\left. + \left(\sum_{l=0}^{L} k_l^{\frac{5}{2}} \Psi_l^{\frac{1}{2}}\right)\left[E_{L+1}^{\frac{1}{2}} \Psi_{L+1}^{\frac{1}{2}} + E_{L+1}^{\frac{1}{2}} \Psi_{L+2}^{\frac{1}{2}} + \sum_{m=L+2}^{\infty} E_m^{\frac{1}{2}}\left(\Psi_{m-1}^{\frac{1}{2}} + \Psi_m^{\frac{1}{2}} + \Psi_{m+1}^{\frac{1}{2}}\right)\right]\right\}. \quad (10)$$

3. Results

Let $\mathbf{v}(\mathbf{x}, t)$ be a finite energy solution of the D-dimensional Navier–Stokes equations (1) with no boundary conditions. Denoting by $\mathscr{E}$ the total energy and by $\hat{\mathbf{v}}(\mathbf{k}, t)$ the D-dimensional Fourier transform of $\mathbf{v}(\mathbf{x}, t)$, we have by Parseval's theorem

$$\mathscr{E} = \frac{1}{2}\int_{\mathbb{R}^D} |\mathbf{v}(\mathbf{x}, t)|^2 d^D x = \frac{1}{2}\int_{\mathbb{R}^D} |\hat{\mathbf{v}}(\mathbf{k}, t)|^2 d^D k.$$

We define the energy spectrum $E(\mathbf{k})$ by

$$E(\mathbf{k}, t) = k^{D-1}|\mathbf{v}(\mathbf{k}, t)|^2.$$

The total energy $\mathscr{E}$ is recovered by integration over the wavenumber k and the angular variables. Notice that $E(\mathbf{k}) \leqslant Ck^{-s}$ implies for the shell-integrated energy $E_n \leqslant C'k_n^{-s+1}$. Similarly, for a passive scalar ϕ satisfying (9), we define

$$I = \frac{1}{2}\int_{\mathbb{R}^D} \phi^2 d^D x = \frac{1}{2}\int_{\mathbb{R}^D} |\hat{\phi}|^2 d^D k$$

and
$$\psi(\mathbf{k}, t) = k^{D-1}|\hat{\phi}(\mathbf{k})|^2.$$

The main results of §2 may now be summarized as follows.

(*a*) *Three-dimensional energy spectrum.* If $E(k) \leqslant Ck^{-s}$ with $s > \frac{8}{3}$, then

$$\lim_{k\to\infty} \Pi_{\text{energy}}(k) = 0,$$

where $\Pi_{\text{energy}}(k)$ is the energy flux through the wavenumber k. For the existence of an inertial range in the limit $\nu \to 0$, it is necessary that

$$\lim_{k\to\infty} \Pi_{\text{energy}}(k) = \epsilon > 0.$$

From the above results, it follows that in the inertial range (if it exists) the spectrum cannot be steeper than $k^{-\frac{8}{3}}$.

It is fairly obvious that the inertial-range energy spectrum cannot be steeper than k^{-3}. Indeed, for $E(k) \leqslant Ck^{-s}$ with $s > 3$, the total enstrophy $\|\text{curl}\,\mathbf{v}\|^2$ is finite; hence the dissipation $\nu\|\text{curl}\,\mathbf{v}\|^2$ goes to zero with the viscosity and cannot cope with a finite energy flux.

(*b*) *Two-dimensional energy spectrum.* If $E(k) \leqslant Ck^{-s}$ with $s > 4$, then

$$\lim_{k\to\infty} \Pi_{\text{enst}}(k) = 0,$$

where $\Pi_{\text{enst}}(k)$ is the enstrophy flux through the wavenumber k. In the enstrophy inertial range (if it exists), the energy spectrum cannot be steeper than k^{-4}.

(*c*) *Passive-scalar advection in inertial range of three-dimensional turbulence.* Using the estimate (10) of §2, we find that, if $E(k) < Ck^{-s}$ with $s \leqslant \frac{8}{3}$ and $\psi(k) < C'k^{-s'}$ with $s' > \frac{1}{2}(8-s)$,

$$\lim_{k\to\infty} \Pi_{\text{scalar}}(k) = 0,$$

where $\Pi_{\text{scalar}}(k)$ is the intensity flux of the passive scalar through the wavenumber k. In particular, if $s = \frac{8}{3}$, the inertial-range spectrum of the passive scalar cannot be steeper than $k^{-\frac{8}{3}}$.

(*d*) *Passive-scalar advection in dissipation range of three-dimensional turbulence.* Using again the estimate (10) of § 2, we find that, if $E(k) < Ce^{-\alpha k}$ and $\psi(k) < C'k^{-s'}$ with $s' > 2$, then

$$\lim_{k\to\infty} \Pi_{\text{scalar}}(k) = 0.$$

In the limit of zero dissipativity κ, the spectrum of the passive scalar in the dissipation range of the turbulence cannot be steeper than k^{-2}.

4. Discussion

First, it is interesting to notice that a $k^{-\frac{8}{3}}$ law for the three-dimensional energy spectrum may be obtained by a Kolmogorov-type dimensional argument. The ingredients of the usual Kolmogorov (1941) argument for *homogeneous* turbulence are (i) a wavenumber k, (ii) the energy density per unit volume at this wavenumber $E_{\text{Kol}}(k)$ and (iii) the dissipation per unit volume ϵ_{Kol}. For the present case of essentially inhomogeneous turbulence of *finite* energy, we could use instead of (ii) and (iii) the total energy density at the wavenumber $E(k)$, which has dimensions $[\text{L}]^6 [\text{T}]^{-2}$, and the total dissipation ϵ, which has dimensions $[\text{L}]^5 [\text{T}]^{-3}$; we then obtain by dimensional analysis

$$E(k) = C\epsilon^{\frac{2}{3}} k^{-\frac{8}{3}}.$$

Carrying out the same analysis for the two-dimensional case yields for the energy spectrum in the enstrophy inertial range

$$E(k) = C'\eta^{\frac{2}{3}} k^{-\frac{11}{3}}$$

(η = total enstrophy dissipation rate), which is not exactly the bound $s = 4$ obtained from the analysis of § 2. The reason for this discrepancy can be readily found by examining the localness of interactions in Fourier space. Let us consider the right-hand side of (6), giving an upper bound for $\Pi_{\text{energy}}(k_L)$ in three dimensions. If we insert $E(k) \sim k^{-\frac{8}{3}}$ (i.e. $E_l \sim k_l^{-\frac{5}{3}}$) we find that the dominant contribution comes from cases where l, m and n are all close to L. If now we take the two-dimensional case [equation (8)], we see that for $E(k) \sim k^{-4}$ (i.e. $E_l \sim k^{-3}$) the dominant contribution comes from l and n close to L and small m, an indication that the enstrophy transfer is strongly non-local (Kraichnan 1971; Pouquet *et al.* 1975).

The following question remains: should the $k^{-\frac{8}{3}}$ law for the three-dimensional case be considered close to the true inertial spectrum for finite energy turbulence? This can hardly be so since we do not expect the small-scale structure of finite energy turbulence to be drastically different from the homogeneous case, which yields *experimentally* a k^{-s} law with s close to $\frac{5}{3}$. Recall that $\frac{8}{3}$ is only an upper bound for the spectral index obtained by a decomposition of Fourier space into shells. As indicated above, interactions in Fourier space appear to be rather local in three dimensions. Still, we have not taken into account another kind of localness, namely in position space: distant fluid elements interact only weakly through the pressure field. This kind of localness is in fact implicitly taken into account in Kolmogorov's dimensional analysis, leading to the $k^{-\frac{5}{3}}$ law, since he considers quantities (energy and dissipation) per *unit volume*. Furthermore, it is not difficult to show that the result of this paper would still hold if in the Navier–Stokes equations the coupling coefficient for each triad of wave vectors

$(\mathbf{k}, \mathbf{p}, \mathbf{q})$ were multiplied by a phase factor $\Phi_{\mathbf{k},\mathbf{p},\mathbf{q}}$ with $\Phi_{-\mathbf{k},-\mathbf{p},-\mathbf{q}} = \Phi^*_{\mathbf{k},\mathbf{p},\mathbf{q}}$ and complete symmetry with respect to **k**, **p** and **q** (similar ideas are found in Kraichnan 1975); under such a modification, any kind of localness in physical space will be entirely lost. This suggests that there is ample room for improvement of the $k^{-\frac{8}{3}}$ estimate.

It is not expected that the spectral index will be decreased from $\frac{8}{3}$ to exactly the value $\frac{5}{3}$ given by Kolmogorov in 1941. Intermittency effects are likely to increase this value, although it is generally believed that the true value is close to $\frac{5}{3}$ (Kolmogorov 1962; Kraichnan 1974; Nelkin 1974). From the present analysis, we can only be certain that it is less than $\frac{8}{3}$. It is interesting to compare our $\frac{8}{3}$ bound with a heuristic result on intermittency obtained by Mandelbrot (1975): if in the limit of very large Reynolds numbers the dissipative structures in the turbulence are concentrated in a set with fractal dimension D, then the inertial-range spectrum should be proportional to $k^{-\frac{1}{3}[5+(3-D)]}$. When dissipation occurs over all space ($D = 3$), this gives a $k^{-\frac{5}{3}}$ law. On the other hand, if dissipation is concentrated in a zero-dimensional set (e.g. isolated points), a $k^{-\frac{8}{3}}$ law is obtained. The true dimension is likely to be greater than two for at least two reasons. First, as noticed by Mandelbrot (1975), the intersection of a D-dimensional set with a line has $D-2$ dimensions if $D \geqslant 2$ and is almost surely empty for $D < 2$, i.e. on moving through the turbulence along a straight line we should not notice its presence. Second, if the pressure and the incompressibility condition are removed, leaving only self-advection, the Navier–Stokes equation becomes a kind of three-dimensional Burgers equation and can be shown to produce shocks, i.e. two-dimensional dissipative structures; pressure-induced instabilities should increase the dimensionality (through twisting and folding of the vortex sheets).

REFERENCES

Bourguignon, J. P. & Brezis, H. 1974 *J. Funct. Anal.* **15**, 341.

Ebin, D. G. & Marsden, J. E. 1970 *Ann. Math.* **92**, 102.

Foias, C., Frisch, U. & Temam, R. 1975 *C. R. Acad. Sci. Paris*, A **280**, 505.

Howard, L. N. 1972 *Ann. Rev. Fluid Mech.* **4**, 473.

Kato, T. 1972 *J. Funct. Anal.* **9**, 296.

Kolmogorov, A. N. 1941 *C. R. Acad. Sci. U.R.S.S.* **30**, 301.

Kolmogorov, A. N. 1962 *J. Fluid Mech.* **13**, 82.

Kraichnan, R. H. 1971 *J. Fluid Mech.* **47**, 525.

Kraichnan, R. H. 1974 *J. Fluid Mech.* **62**, 305.

Kraichnan, R. H. 1975 *Adv. in Math.* **16**, 305.

Ladyzhenskaya, O. A. 1969 *The Mathematical Theory of Viscous Incompressible Flow*, 2nd English edn. Gordon & Breach.

Lichtenstein, L. 1925 *Mathematische Zeitschrift*, p. 89. Springer.

Lions, J. L. 1969 *Quelques Methodes de Resolution des Problèmes aux Limites Non-linéaires.* Dunod-Gauthier-Villars.

Mandelbrot, B. B. 1975 Some fractal aspects of turbulence: intermittency, dimension, kurtosis, and the spectral exponent $5/3+B$. *Proc. Journées Mathématiques sur la Turbulence, Orsay* (ed. R. Temam). Springer.

Nelkin, M. 1974 *Phys. Rev.* A**9**, 388.

Pouquet, A., Lesieur, M., André, J. C. & Basdevant, C. 1975 *J. Fluid Mech.* **72**, 305.

Temam, R. 1975 *J. Funct. Anal.* **20**, 32.